

IDEAL PRECONDITIONERS FOR SADDLE POINT SYSTEMS WITH A RANK-DEFICIENT LEADING BLOCK*

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Abstract. We consider the iterative solution of symmetric saddle point systems with a rank-deficient leading block. We develop two preconditioners that, under certain assumptions on the rank structure of the system, yield a preconditioned matrix with a constant number of eigenvalues. We then derive some properties of the inverse of a particular class of saddle point system and exploit these to develop a third preconditioner, which remains ideal even when the earlier assumptions on rank structure are relaxed.

1. Introduction. Consider the saddle point linear system

$$(1.1) \quad \underbrace{\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix}}_{=: \mathcal{K}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix},$$

where $A \in \mathbb{R}^{n \times n}$ is symmetric positive semidefinite, $B \in \mathbb{R}^{m \times n}$, and $m \leq n$. We assume throughout this report that B has full row rank (a necessary condition for \mathcal{K} to be nonsingular).

When A is positive definite, certain block Schur complement preconditioners yield preconditioned matrices with a constant number of eigenvalues [7]. Our goal in this report is to develop such ideal preconditioners when A is singular. A common method for dealing with a singular A is the *augmented Lagrangian* approach (see, e.g., [5]), which replaces A with a positive definite augmented leading block of the form

$$A + B^T W^{-1} B,$$

where W is a positive definite weight matrix. Our approach here is similar, but rather than augment with the entire matrix B , we will use only as many rows of B as are necessary to alleviate the rank-deficiency of A .

We will see that we can sometimes develop preconditioners that are not ideal in general, but are ideal under certain assumptions on the rank structure of \mathcal{K} . Here we define two such assumptions that will be relevant in the upcoming discussion.

DEFINITION 1.1 (Maximal rank-deficiency). *Let \mathcal{K} be a nonsingular symmetric saddle point matrix with $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{m \times n}$. We say that \mathcal{K} has a maximally rank-deficient leading block if $\text{null}(A) = m$.*

Because \mathcal{K} is nonsingular if and only if $\ker(A) \cap \ker(B) = \emptyset$ [1, Theorem 3.2], having the nullity of A be greater than m would necessarily make \mathcal{K} singular – hence the term “maximally rank-deficient.” Estrin and Greif [2] develop preconditioners for matrices with this property.

DEFINITION 1.2 (Minimal independence). *Let \mathcal{K} be a nonsingular symmetric saddle point matrix whose leading block A has nullity $m_2 = m - m_1$, for $m_1 \geq 0$. We say that \mathcal{K} has minimally independent off-diagonal blocks if B contains m_2 row vectors outside the row space of A and m_1 vectors within the row space. We write this as a*

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3×3 blocking of \mathcal{K} :

$$(1.2) \quad \mathcal{K} = \begin{bmatrix} A & B_1^T & B_2^T \\ B_1 & 0 & 0 \\ B_2 & 0 & 0 \end{bmatrix},$$

where the rows of $B_1 \in \mathbb{R}^{m_1 \times n}$ are in the row space of A and those of $B_2 \in \mathbb{R}^{m_2 \times n}$ are outside the row space of A .

Assuming that B_1 contains the rows that are linearly dependent on A incurs no loss of generality because, for any \mathcal{K} that satisfies the minimal independence condition, we can permute B so that this happens. We call this “minimal independence” because if there were any fewer linearly independent rows in B_2 , then \mathcal{K} would be singular. Matrices with a maximally rank-deficient leading block are a special case, with m_1 (the number of rows in the dependent block B_1) being zero. Notice also that this definition guarantees that the submatrix

$$\begin{bmatrix} A & B_2^T \\ B_2 & 0 \end{bmatrix}$$

is invertible.

In [section 2](#), we present two preconditioners that are ideal in the cases of maximal rank-deficiency of A and minimal independence of B . [Section 3](#) explores some unique properties of matrices with minimally independent B , which we then exploit in [section 4](#) to develop a 3×3 -block preconditioner that remains ideal even when the minimal independence assumption no longer holds. [Table 1.1](#) summarizes the three preconditioners presented in this report.

	Preconditioner		
	$\mathcal{P}_{2,D}$	$\mathcal{P}_{3,D}$	$\mathcal{P}_{3,T}$
Conditions under which \mathcal{P} is ideal			
Assumed null(A):	m	less than m	less than m
Assumptions on B :	none	minimal independence	none
Properties of \mathcal{P}			
\mathcal{P} positive definite?	yes	yes	no
Need to split B ?	no	yes	yes
Need ker(A) basis?	no	no	yes (or ker(B_2))
Described in:	Subsection 2.1 (eq. (2.2))	Subsection 2.2 (eq. (2.4))	Section 4 (eq. (4.2))

TABLE 1.1

Summary of preconditioners for nonsingular saddle point systems with a singular leading block, where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{m \times n}$.

2. Preconditioners for special cases.

2.1. A preconditioner for maximal rank-deficiency. We assume here that $\text{null}(A) = m$ (the number of rows in B). This means we must augment the leading block with all the rows of B in order to make it full rank. For a positive definite weight matrix $W_B \in \mathbb{R}^{m \times m}$, we define the augmented system $\mathcal{K}_2(W_B)$ by

$$(2.1) \quad \mathcal{K}_2(W_B) := \begin{bmatrix} A + B^T W_B^{-1} B & B^T \\ B & 0 \end{bmatrix}.$$

We recall the following result about the Schur complement of the leading block in $\mathcal{K}_2(W_B)$ (see [2, Theorem 3.5]):

PROPOSITION 2.1. *Suppose $\text{null}(A) = m$ and let $W_B \in \mathbb{R}^{m \times m}$ be an invertible matrix. Then*

$$B(A + B^T W_B^{-1} B)^{-1} B^T = W_B.$$

Motivated by this result, we precondition \mathcal{K} by taking the block diagonal Schur complement preconditioner [7, equation (2)] of $\mathcal{K}_2(W_B)$, given by

$$(2.2) \quad \mathcal{P}_{2,D} := \begin{bmatrix} A + B^T W_B^{-1} B & 0 \\ 0 & W_B \end{bmatrix}.$$

The 2 in the subscript refers to the block size of the matrix and D refers to the fact that the preconditioner is block diagonal. This particular preconditioner is also described in [6].

THEOREM 2.2. *Suppose $\text{null}(A) = m$ and let $W_B \in \mathbb{R}^{m \times m}$ be a positive definite matrix. The preconditioned matrix $\mathcal{P}_{2,D}^{-1} \mathcal{K}$ has eigenvalues $\lambda = 1, -1$, with respective geometric multiplicities n and m .*

Proof. The generalized eigenvalue problem is

$$(2.3a) \quad Ax + B^T y = \lambda Ax + \lambda B^T W_B^{-1} Bx;$$

$$(2.3b) \quad Bx = \lambda W_B y.$$

We immediately obtain n eigenvectors by observing that (2.3) is satisfied for $\lambda = 1$ by vectors of the form

$$\begin{bmatrix} x \\ W_B^{-1} Bx \end{bmatrix}, \quad x \in \mathbb{R}^n.$$

Moreover, any vector of the form

$$\begin{bmatrix} x \\ -W_B^{-1} Bx \end{bmatrix}, \quad x \in \ker(A),$$

satisfies (2.3) for $\lambda = -1$. Because $\text{null}(A) = m$, we have accounted for all $n + m$ eigenvectors of $\mathcal{P}_{2,D}^{-1} \mathcal{K}$. \square

This preconditioner can be used when A is not maximally rank-deficient, though it will no longer be ideal. We refer the reader to [6] for analysis of this case.

2.2. A preconditioner for minimal independence. We now assume that B is minimally independent and that the m rows of B have been partitioned as in (1.2) –i.e., that with $\text{null}(A) = m_2$,

$$\mathcal{K} = \begin{bmatrix} A & B_1^T & B_2^T \\ B_1 & 0 & 0 \\ B_2 & 0 & 0 \end{bmatrix},$$

where the rows of $B_1 \in \mathbb{R}^{m_1 \times n}$ are linearly dependent on the rows of A and those of $B_2 \in \mathbb{R}^{m_2 \times n}$ are linearly independent.

The matrix $A + B_2^T W^{-1} B_2$ will be positive definite for any positive definite weight matrix $W \in \mathbb{R}^{m_2 \times m_2}$ (we omit the subscript to distinguish this from the $m \times m$ weight

matrix W_B in the previous subsection). We adapt the preconditioner $\mathcal{P}_{2,D}$ to this case by defining the 3×3 block diagonal preconditioner:

$$(2.4) \quad \mathcal{P}_{3,D} = \begin{bmatrix} A + B_2^T W^{-1} B_2 & 0 & 0 \\ 0 & (B_1(A + B_2^T W^{-1} B_2)^{-1} B_1^T)/2 & 0 \\ 0 & 0 & W \end{bmatrix}.$$

When $m_1 = 0$ (i.e., B contains no rows that depend on A), this preconditioner reduces to $\mathcal{P}_{2,D}$. We can view this preconditioner as applying $\mathcal{P}_{2,D}$ to the subsystem $\begin{bmatrix} A & B_2^T \\ B_2 & 0 \end{bmatrix}$ and then handling the additional block B_1 with a Schur complement of the augmented leading block. We will show that the scaling factor of $\frac{1}{2}$ for this Schur complement leads to fewer eigenvalues of the preconditioned operator.

2.3. Spectral analysis for the minimally independent case. Before presenting the spectral analysis of $\mathcal{P}_{3,D}^{-1} \mathcal{K}$, we establish some lemmas that will prove helpful. We begin by recalling the following result [3, Corollary 2.1]:

LEMMA 2.3. *For matrices $M, N \in \mathbb{R}^{n \times n}$, if $\text{rank}(M) = r$, $\text{rank}(N) = n - r$, and $\text{rank}(M + N) = n$, then $(M + N)^{-1} M$ is a projector of rank r .*

When B is minimally independent, $\text{rank}(A) = n - m_2$ and $\text{rank}(B_2^T W^{-1} B_2) = m_2$. Therefore, we can use Lemma 2.3 to establish the following:

LEMMA 2.4. *Suppose that $\text{null}(A) = m_2$, B_2 is a subset of m_2 rows of B such that $\begin{bmatrix} A & B_2^T \\ B_2 & 0 \end{bmatrix}$ is invertible, and W is positive definite. Then $(A + B_2^T W^{-1} B_2)^{-1} B_2^T$ is a null matrix of A .*

Proof. We can write

$$\begin{aligned} A(A + B_2^T W^{-1} B_2)^{-1} B_2^T &= (A + B_2^T W^{-1} B_2 - B_2^T W^{-1} B_2)(A + B_2^T W^{-1} B_2)^{-1} B_2^T \\ &= B_2^T - B_2^T W^{-1} B_2 (A + B_2^T W^{-1} B_2)^{-1} B_2^T. \end{aligned}$$

By Lemma 2.3, $(A + B_2^T W^{-1} B_2)^{-1} B_2^T W^{-1} B_2$ is a projector of rank m_2 . Because

$$B_2^T W^{-1} B_2 (A + B_2^T W^{-1} B_2)^{-1} = \left((A + B_2^T W^{-1} B_2)^{-1} B_2^T W^{-1} B_2 \right)^T,$$

$B_2^T W^{-1} B_2 (A + B_2^T W^{-1} B_2)^{-1}$ is also a projector of rank m_2 . Moreover, it is clear that its range is a subset of $\text{range}(B_2^T)$. Because both B_2^T and $B_2^T W^{-1} B_2 (A + B_2^T W^{-1} B_2)^{-1}$ have rank m_2 , we conclude that the projector's range is in fact equal to $\text{range}(B_2^T)$. Therefore,

$$\begin{aligned} A(A + B_2^T W^{-1} B_2)^{-1} B_2^T &= B_2^T - B_2^T W^{-1} B_2 (A + B_2^T W^{-1} B_2)^{-1} B_2^T \\ &= B_2^T - B_2^T \\ &= 0. \end{aligned} \quad \square$$

This lets us make a more specific observation about these projectors.

LEMMA 2.5. *Suppose that $\text{null}(A) = m_2$, B_2 is a subset of m_2 rows of B such that $\begin{bmatrix} A & B_2^T \\ B_2 & 0 \end{bmatrix}$ is invertible, and W is positive definite. Then*

$$A(A + B_2^T W^{-1} B_2)^{-1} =: P_A$$

is a projector onto $\text{range}(A)$ along $\ker(B_2)$.

Proof. Because [Lemma 2.3](#) tells us that P_A^T is a projector of rank $n - m_2$, we deduce that P_A is as well. Notice that P_A is equal to A multiplied by a full-rank matrix, which means that $\text{range}(P_A) = \text{range}(A)$.

Because $\text{null}(P_A) = m_2$, showing that the projection is along $\ker(B_2)$ is equivalent to showing that $P_A B_2^T = 0$; this follows from [Lemma 2.4](#). \square

With this last result established, we can state our main result for this subsection.

THEOREM 2.6. *Suppose that $\text{null}(A) = m_2$, with B minimally independent and partitioned as in (1.2), and let W be positive definite. The preconditioned matrix $\mathcal{P}_{3,D}^{-1}\mathcal{K}$ has eigenvalues $\lambda = -1, 1, 2$, with respective geometric multiplicities $m_1 + m_2$, $n - m_1$, and m_1 .*

Proof. Define $\tilde{A}_W := A + B_2^T W^{-1} B_2$ and $S_B := B_1 \tilde{A}_W^{-1} B_1^T$. The eigenvalues of $\mathcal{P}_{3,D}^{-1}\mathcal{K}$ are given by the generalized eigenvalue problem:

$$(2.5a) \quad Ax + B_1^T y_1 + B_2^T y_2 = \lambda \tilde{A}_W x;$$

$$(2.5b) \quad B_1 x = \frac{\lambda}{2} S_B y_1;$$

$$(2.5c) \quad B_2 x = \lambda W y_2.$$

Obtaining $y_1 = \frac{2}{\lambda} S_B^{-1} B_1 x$ from (2.5b), $y_2 = \frac{1}{\lambda} W^{-1} B_2 x$ from (2.5c), substituting these values into (2.5a), and multiplying both sides from the left by \tilde{A}_W^{-1} gives:

$$\tilde{A}_W^{-1} Ax + \frac{2}{\lambda} \tilde{A}_W^{-1} B_1^T S_B^{-1} B_1 x + \frac{1}{\lambda} \tilde{A}_W^{-1} B_2^T W^{-1} B_2 x = \lambda x.$$

Because $\tilde{A}_W^{-1}(A + B_2^T W^{-1} B_2) = I$, we replace $\tilde{A}_W^{-1} B_2^T W^{-1} B_2$ by $I - \tilde{A}_W^{-1} A$. We then multiply all terms by λ and rearrange to give:

$$(2.6) \quad \lambda^2 x - \lambda \underbrace{\tilde{A}_W^{-1} A}_{=: P_1} x - \left(\underbrace{(I - \tilde{A}_W^{-1} A)}_{=: I - P_1} + 2 \underbrace{\tilde{A}_W^{-1} B_1^T S_B^{-1} B_1}_{=: P_2} \right) x = 0.$$

[Lemma 2.3](#) shows that P_1 is a projector. Consequently, $I - P_1$ is as well. Note also that $P_2 = \tilde{A}_W^{-1} B_1^T S_B^{-1} B_1$ is a projector onto the range of $\tilde{A}_W^{-1} B_1^T$. We can now show the desired geometric multiplicities by considering x in the ranges (or null spaces) of the projectors in (2.6).

First, consider $x \in \text{range}(\tilde{A}_W^{-1} B_1^T)$. Clearly, $P_2 x = x$. And because all columns of B_1^T are in $\text{range}(A)$,

$$(2.7) \quad \text{range}(P_2) = \text{range}(\tilde{A}_W^{-1} B_1^T) \subseteq \text{range}(\tilde{A}_W^{-1} A) = \text{range}(P_1).$$

Thus, $P_1 x = x$ and $(I - P_1)x = 0$, so (2.6) becomes $\lambda^2 x - \lambda x - 2x = 0$. This yields the eigenvalues -1 and 2 , each with geometric multiplicity m_1 .

Next, consider $x \in \ker(A)$. Clearly, $P_1 x = 0$ and $(I - P_1)x = x$. Notice that because the rows of B_1 are in the row space of A , we can write $B_1 = QA$ for $Q \in \mathbb{R}^{m_1 \times n}$. Thus,

$$(2.8) \quad \ker(A) \subseteq \ker(B_1),$$

so $P_2 x = 0$. [Equation \(2.6\)](#) then becomes $\lambda^2 x - x = 0$, which yields $\lambda = \pm 1$, each with geometric multiplicity m_2 .

Lastly, let us consider the term $\left((I - P_1) + P_2\right)$. Notice that

$$\text{range}(I - P_1) = \ker(P_1) \subseteq \ker(P_2),$$

where the second relation holds because of (2.8). Thus, the range of $(I - P_1)$ does not overlap with that of P_2 , so

$$\text{rank}\left((I - P_1) + 2P_2\right) = \text{rank}(I - P_1) + \text{rank}(P_2) = m_2 + m_1.$$

Another consequence of the lack of overlap is that $\left((I - P_1) + 2P_2\right)x = 0$ if and only if $(I - P_1)x = 0$ and $P_2x = 0$. Therefore, any $x \in \ker\left((I - P_1) + 2P_2\right)$ also satisfies $P_1x = x$, which means that (2.6) becomes $\lambda^2x - \lambda x = 0$. Because \mathcal{K} is nonsingular, $\lambda = 0$ cannot be an eigenvalue; we therefore have $n - m_1 - m_2$ additional eigenvectors corresponding to $\lambda = 1$.

We have now accounted for all $n + m_1 + m_2$ eigenvectors, which completes the proof. \square

Remark 2.7. We saw that, by scaling the $(2, 2)$ -block of $\mathcal{P}_{3,D}$ by $\frac{1}{2}$, we obtain a preconditioned operator with three eigenvalues. With minor modifications to the previous analysis, it is easy to show that other scaling factors yield four eigenvalues. In terms of the number of eigenvalues, there is no benefit in scaling the $(1, 1)$ - or $(3, 3)$ -blocks.

3. Block 3×3 inverse formulas for the saddle point matrix. Estrin and Greif [2] showed that when A is maximally rank-deficient, the inverse of \mathcal{K} has some surprising properties. They then exploited these to develop a block triangular approximate inverse preconditioner. In this section we show that analogous properties hold when B is minimally independent; indeed, recalling that maximal rank-deficiency of A is a special case of minimal independence of B , many of the results in [2] can be re-derived as corollaries of the results presented here. Later (section 4), we derive a block triangular preconditioner based on these insights. While the resulting preconditioner is more expensive than the block diagonal preconditioners of section 2, it is more generalizable: it remains ideal for systems in which A is not maximally rank-deficient and B is not minimally independent.

We begin from two known expressions for the inverse of the standard 2×2 saddle point system,

$$K_2 = \begin{bmatrix} \mathcal{A} & \mathcal{B}^T \\ \mathcal{B} & 0 \end{bmatrix}.$$

When \mathcal{A} is invertible, we can write (see [1, equation (3.4)])

$$(3.1) \quad K_2^{-1} = \begin{bmatrix} \mathcal{A}^{-1} - \mathcal{A}^{-1}\mathcal{B}^T S_{\mathcal{B}}^{-1}\mathcal{B}\mathcal{A}^{-1} & \mathcal{A}^{-1}\mathcal{B}^T S_{\mathcal{B}}^{-1} \\ S_{\mathcal{B}}^{-1}\mathcal{B}\mathcal{A}^{-1} & -S_{\mathcal{B}}^{-1} \end{bmatrix},$$

where $S_{\mathcal{B}} = \mathcal{B}\mathcal{A}^{-1}\mathcal{B}^T$. A second formula [1, equation (3.8)] does not require \mathcal{A} to be invertible, and instead makes use of a matrix $Z_{\mathcal{B}} \in \mathbb{R}^{n \times (n-m)}$ whose columns form a basis for the null space of \mathcal{B} :

$$(3.2) \quad K_2^{-1} = \begin{bmatrix} V_{\mathcal{B}} & (I - V_{\mathcal{B}}\mathcal{A})\mathcal{B}^T(\mathcal{B}\mathcal{B}^T)^{-1} \\ (\mathcal{B}\mathcal{B}^T)^{-1}\mathcal{B}(I - \mathcal{A}V_{\mathcal{B}}) & -(\mathcal{B}\mathcal{B}^T)^{-1}\mathcal{B}(\mathcal{A} - \mathcal{A}V_{\mathcal{B}}\mathcal{A})\mathcal{B}^T(\mathcal{B}\mathcal{B}^T)^{-1} \end{bmatrix},$$

where $V_{\mathcal{B}} = Z_{\mathcal{B}}(Z_{\mathcal{B}}^T\mathcal{A}Z_{\mathcal{B}})^{-1}Z_{\mathcal{B}}^T$.

3.1. Null- B_2 inverse formula. We now return our attention to the system \mathcal{K} . Assume that B is minimally independent and split as in (1.2). We then partition \mathcal{K} as

$$\mathcal{K} := \left[\begin{array}{cc|c} A & B_1^T & B_2^T \\ B_1 & 0 & 0 \\ \hline B_2 & 0 & 0 \end{array} \right],$$

and treat it as a 2×2 block system with a singular leading block. We apply formula (3.2) with

$$\mathcal{A} = \begin{bmatrix} A & B_1^T \\ B_1 & 0 \end{bmatrix}, \quad \mathcal{B} = [B_2 \quad 0].$$

For the null matrix $Z_{\mathcal{B}}$ of \mathcal{B} , we use

$$Z_{\mathcal{B}} = \begin{bmatrix} Z_{B_2} & 0 \\ 0 & I \end{bmatrix},$$

where $Z_{B_2} \in \mathbb{R}^{n \times (n-m_2)}$ is a matrix whose columns form a basis for the null space of B_2 and I denotes the identity matrix of size m_1 . Computing $V_{\mathcal{B}}$ requires the inverse of

$$Z_{\mathcal{B}}^T \mathcal{A} Z_{\mathcal{B}} = \begin{bmatrix} Z_{B_2}^T A Z_{B_2} & Z_{B_2}^T B_1^T \\ B_1 Z_{B_2} & 0 \end{bmatrix}.$$

Because the rows of B_2 are outside the row space of A , the matrix $\begin{bmatrix} A & B_2^T \\ B_2 & 0 \end{bmatrix}$ is nonsingular. This means that $Z_{B_2}^T A Z_{B_2}$ is positive definite [1, p. 20], and that $Z_{\mathcal{B}}^T \mathcal{A} Z_{\mathcal{B}}$ is a saddle point system with a positive definite leading block. Thus, we can compute its inverse using (3.1) and substitute this back into formula (3.2) to obtain

$$(3.3a) \quad \mathcal{K}^{-1} = \begin{bmatrix} X_1 & X_2^T & X_3^T \\ X_2 & X_4 & X_5^T \\ X_3 & X_5 & X_6 \end{bmatrix},$$

with

$$(3.3b) \quad X_1 := V - V B_1^T S_V^{-1} B_1 V$$

$$(3.3c) \quad X_2 := S_V^{-1} B_1 V$$

$$(3.3d) \quad X_3 := (B_2 B_2^T)^{-1} B_2 \left(I - AV + AV B_1^T S_V^{-1} B_1 V - B_1^T S_V^{-1} B_1 V \right)$$

$$(3.3e) \quad X_4 := -S_V^{-1}$$

$$(3.3f) \quad X_5 := (B_2 B_2^T)^{-1} B_2 (I - AV) B_1^T S_V^{-1}$$

$$(3.3g) \quad X_6 := -(B_2 B_2^T)^{-1} B_2 \left(A + B_1^T S_V^{-1} B_1 - AVA - AV B_1^T S_V^{-1} B_1 - \right. \\ \left. B_1^T S_V^{-1} B_1 VA + AV B_1^T S_V^{-1} B_1 VA \right) B_2^T (B_2 B_2^T)^{-1},$$

where $V := Z_{B_2} (Z_{B_2}^T A Z_{B_2})^{-1} Z_{B_2}^T$ and $S_V := B_1 V B_1^T$.

Equation (3.3) holds for any symmetric nonsingular \mathcal{K} such that $\begin{bmatrix} A & B_2^T \\ B_2 & 0 \end{bmatrix}$ is invertible. But fortunately for us, the assumption that B is minimally independent lets us simplify some of the uglier terms. We begin with another lemma featuring the projection matrix P_A of Lemma 2.5.

LEMMA 3.1. *If $\text{null}(A) = m_2$ and B_2 is a subset of m_2 rows of B such that $\begin{bmatrix} A & B_2^T \\ B_2 & 0 \end{bmatrix}$ is invertible, then $AV = P_A$, where P_A is a projector onto $\text{range}(A)$ along $\ker(B_2)$.*

Proof. By writing

$$AV = AZ_{B_2}(Z_{B_2}^T AZ_{B_2})^{-1} Z_{B_2}^T,$$

we see that AV is a projector onto $\text{range}(AZ_{B_2})$ along $\ker(B_2)$. It is clear that $\text{range}(AZ_{B_2}) \subseteq \text{range}(A)$; moreover, A has rank $n - m_2$ and AZ_{B_2} has rank $n - m_2$. Because the range of AZ_{B_2} is a subset of the range of A , and both ranges have the same dimension, we conclude that $\text{range}(AZ_{B_2}) = \text{range}(A)$. \square

THEOREM 3.2. *Let \mathcal{K} have minimally independent off-diagonal blocks, with B partitioned as in (1.2). Then the (2,3)-, (3,3)- and (3,2)-blocks of \mathcal{K}^{-1} are equal to zero, i.e.:*

$$\mathcal{K}^{-1} = \begin{bmatrix} \times & \times & \times \\ \times & \times & 0 \\ \times & 0 & 0 \end{bmatrix}.$$

Proof. We begin with the (2,3)-block of \mathcal{K}^{-1} , denoted by X_5 (eq. (3.3f)). Because the columns of B_1^T are in the range of A , Lemma 3.1 implies that $AVB_1^T = B_1^T$, which gives $(I - AV)B_1^T = 0$. Hence, $X_5 = 0$.

Now consider the (3,3)-block, X_6 (eq. (3.3g)). Using $AVA = A$ and $AVB_1^T = B_1^T$, we find that the middle multiplicative term of X_6 (between the large parentheses) simplifies to zero, giving $X_6 = 0$. \square

We can also simplify the (3,1)-block.

PROPOSITION 3.3. *Let \mathcal{K} be nonsingular with minimally independent off-diagonal blocks, with B partitioned as in (1.2). Then the (1,3)- and (3,1)-blocks of \mathcal{K}^{-1} can be simplified by writing*

$$X_3 = (B_2 B_2^T)^{-1} B_2 (I - AV).$$

Proof. The result follows from the expression for X_3 in (3.3d) and the observation in the proof of Theorem 3.2 that $AVB_1^T = B_1^T$. \square

Incorporating the results of Theorem 3.2 and Proposition 3.3 into the original inverse formula (3.3), we obtain the following simplified formula when \mathcal{K} has minimally independent off-diagonal blocks:

$$(3.4) \quad \mathcal{K}^{-1} = \begin{bmatrix} V - VB_1^T S_V^{-1} B_1 V & VB_1^T S_V^{-1} & (I - VA)B_2^T (B_2 B_2^T)^{-1} \\ S_V^{-1} B_1 V & -S_V^{-1} & 0 \\ (B_2 B_2^T)^{-1} B_2 (I - AV) & 0 & 0 \end{bmatrix}.$$

Because this expression uses a null space of B_2 in computing V , we refer to it as the *null- B_2 inverse formula*.

3.2. Null- A inverse formula. The zero blocks in the inverse of \mathcal{K} let us derive a simpler expression for \mathcal{K}^{-1} . We recall the following result [4, Proposition 2.1]:

PROPOSITION 3.4. *Define*

$$K_2 = \begin{bmatrix} \mathcal{A} & \mathcal{B}^T \\ \mathcal{B} & 0 \end{bmatrix} \quad \text{and} \quad K_2(W) = \begin{bmatrix} \mathcal{A} + \mathcal{B}^T W^{-1} \mathcal{B} & \mathcal{B}^T \\ \mathcal{B} & 0 \end{bmatrix},$$

where $\mathcal{A} \in \mathbb{R}^{n \times n}$, $\mathcal{B} \in \mathbb{R}^{m \times n}$, and $W \in \mathbb{R}^{m \times m}$. If K_2 and $K_2(W)$ are both invertible, then

$$(K_2(W))^{-1} = K_2^{-1} - \begin{bmatrix} 0 & 0 \\ 0 & W^{-1} \end{bmatrix}.$$

Therefore, when we augment the leading block of \mathcal{K} with B_2 , we obtain

$$(3.5) \quad \underbrace{\begin{bmatrix} A + B_2^T W^{-1} B_2 & B_1^T & B_2^T \\ B_1 & 0 & 0 \\ B_2 & 0 & 0 \end{bmatrix}^{-1}}_{=: \mathcal{K}(W)} = \mathcal{K}^{-1} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & W^{-1} \end{bmatrix}.$$

If W is positive definite, then $A + B_2^T W^{-1} B_2$ is positive definite. We can obtain the inverse of $\mathcal{K}(W)$ by partitioning the matrix into a 2×2 system and applying formula (3.1), or we can compute it directly by Gaussian elimination. The result is

$$(\mathcal{K}(W))^{-1} = \begin{bmatrix} \hat{A} - \hat{A} B_2^T \bar{S}^{-1} B_2 \hat{A} & (I - \hat{A} B_2^T \bar{S}^{-1} B_2) \tilde{A}_W^{-1} B_1^T S_B^{-1} & \hat{A} B_2^T \bar{S}^{-1} \\ S_B^{-1} B_1 \tilde{A}_W^{-1} (I - B_2^T \bar{S}^{-1} B_2 \hat{A}) & -S_B^{-1} - S_B^{-1} B_1 \tilde{A}_W^{-1} B_2^T \bar{S}^{-1} B_2 \tilde{A}_W^{-1} B_1^T S_B^{-1} & S_B^{-1} B_1 \tilde{A}_W^{-1} B_2^T \bar{S}^{-1} \\ \bar{S}^{-1} B_2 \hat{A} & \bar{S}^{-1} B_2 \tilde{A}_W^{-1} B_1^T S_B^{-1} & -\bar{S}^{-1} \end{bmatrix},$$

where $\tilde{A}_W = A + B_2^T W^{-1} B_2$, $S_B := B_1 \tilde{A}_W^{-1} B_1^T$, $\hat{A} := \tilde{A}_W^{-1} - \tilde{A}_W^{-1} B_1^T S_B^{-1} B_1 \tilde{A}_W^{-1}$, and $\bar{S} := B_2 \hat{A} B_2^T$. This formula is valid for any 3×3 saddle point system with an invertible leading block \tilde{A}_W .

When B is minimally independent, we can combine (3.5) with the result of [Theorem 3.2](#) that the (3,3)-block of \mathcal{K}^{-1} is zero to conclude that

$$(3.6) \quad \bar{S} = W.$$

It turns out that some of the terms of $(\mathcal{K}(W))^{-1}$ are related to the null space of A and that we can further simplify our inverse formula using these connections. First, recall from (2.8) that $\ker(A) \subseteq \ker(B_1)$. This lets us establish the following simplifying result:

PROPOSITION 3.5. *Let \mathcal{K} be nonsingular with minimally independent off-diagonal blocks, with B partitioned as in (1.2). Then $\tilde{A}_W^{-1} B_2^T = \hat{A} B_2^T$ for any positive definite W .*

Proof. We can write

$$\begin{aligned} \hat{A} B_2^T &= \tilde{A}_W^{-1} B_2^T - \tilde{A}_W^{-1} B_1^T S_B^{-1} B_1 \tilde{A}_W^{-1} B_2^T \\ &= (I - \tilde{A}_W^{-1} B_1^T S_B^{-1} B_1) \tilde{A}_W^{-1} B_2^T. \end{aligned}$$

Notice that $\tilde{A}_W^{-1} B_1^T S_B^{-1} B_1$ is a projector onto $\text{range}(\tilde{A}_W^{-1} B_1^T)$ along $\text{range}(B_1^T)$. Therefore, $(I - \tilde{A}_W^{-1} B_1^T S_B^{-1} B_1)$ is a projector onto $\ker(B_1)$. By [Lemma 2.4](#), $\tilde{A}_W^{-1} B_2^T$ is a null matrix of A . Thus, it is also a null matrix of B_1 , and is therefore unchanged by the projection. \square

So far, all results presented hold for any positive definite W . The following proposition – which is essentially the converse of [Lemma 2.4](#) – shows how one specific choice of W can lead to a simpler expression for \mathcal{K}^{-1} .

PROPOSITION 3.6. *Let $Z_A \in \mathbb{R}^{n \times m_2}$ be a matrix whose columns form a basis of $\ker(A)$. This matrix can be written in the form*

$$Z_A = \tilde{A}_W^{-1} B_2^T$$

with $W = B_2 Z_A$.

Proof. We first show that $B_2 Z_A$ is nonsingular. Assume that

$$(B_2 Z_A)x = B_2(Z_A x) = 0$$

for some x . Clearly, $Z_A x \in \ker(A)$. Moreover, $B_2 Z_A x = 0$ implies that $Z_A x \in \ker(B_2)$. But, we assumed that $\begin{bmatrix} A & B_2^T \\ B_2 & 0 \end{bmatrix}$ is nonsingular, and the existence of a nonzero vector $Z_A x$ belonging to both null spaces would contradict this assumption. Therefore, x must be zero, which means that $B_2 Z_A$ is nonsingular.

Next, the matrix Z_A is equal to $\tilde{A}_W^{-1} B_2^T$ if and only if $\tilde{A}_W Z_A = B_2^T$. We now confirm that when $W = B_2 Z_A$,

$$\tilde{A}_W Z_A = (A + B_2^T (B_2 Z_A)^{-1} B_2) Z_A = B_2^T$$

for any Z_A such that $A Z_A = 0$, as required. \square

We can now use the previous results to simplify the expression for \mathcal{K}^{-1} . Let us denote the weight matrix by

$$(3.7) \quad L := B_2 Z_A$$

to distinguish it from the arbitrary weight matrix W . We assume without loss of generality that L is symmetric positive definite. (We say ‘‘without loss of generality’’ because we can always make L positive definite by taking $Z_A (B_2 Z_A)^T$ as our null matrix of A , and defining $L = B_2 Z_A (B_2 Z_A)^T$.) We drop the subscript from the corresponding augmented matrix \tilde{A}_L and define

$$\tilde{A} := \tilde{A}_L = A + B_2^T L^{-1} B_2.$$

We can simplify the expression for $(\mathcal{K}(W))^{-1}$ as follows: replace all occurrences of \tilde{S} with L (Equation (3.6)); $\tilde{A}^{-1} B_2^T$ or $\tilde{A} B_2^T$ with Z_A (Propositions 3.5 and 3.6); and $B_1 Z_A$ or $Z_A^T B_1^T$ with zero (Equation (2.8)). This gives

$$(3.8) \quad \mathcal{K}^{-1} = \begin{bmatrix} \hat{A} - Z_A L^{-1} Z_A^T & \tilde{A}^{-1} B_1^T S_B^{-1} & Z_A L^{-1} \\ S_B^{-1} B_1 \tilde{A}^{-1} & -S_B^{-1} & 0 \\ L^{-1} Z_A^T & 0 & 0 \end{bmatrix}.$$

We refer to equation (3.8) as the *null-A inverse formula*. Recalling that $Z_A = \tilde{A}^{-1} B_2^T$, we can alternatively write the leading block in multiplicative form:

$$(3.9) \quad \mathcal{K}^{-1} = \begin{bmatrix} (I - \tilde{A}^{-1} B_1^T S_B^{-1} B_1) \tilde{A}^{-1} (I - B_2^T L^{-1} Z_A^T) & \tilde{A}^{-1} B_1^T S_B^{-1} & Z_A L^{-1} \\ S_B^{-1} B_1 \tilde{A}^{-1} & -S_B^{-1} & 0 \\ L^{-1} Z_A^T & 0 & 0 \end{bmatrix}.$$

4. A preconditioner for the general case. In this section, we propose a block triangular preconditioner based on our insights about \mathcal{K}^{-1} from section 3. Though these analyses assumed that B was minimally independent, we will combine elements of the null- B_2 and null- A formulas to obtain a preconditioner that remains ideal even when this assumption does not hold. While we continue to assume that B is partitioned into $B_1 \in \mathbb{R}^{m_1 \times n}$ and $B_2 \in \mathbb{R}^{m_2 \times n}$, and that $\begin{bmatrix} A & B_2^T \\ B_2 & 0 \end{bmatrix}$ is nonsingular, we no longer assume that the rows of B_1 are in the row space of A .

The following result illustrates a connection between the null- B_2 and null- A formulas.

PROPOSITION 4.1. *Suppose that $\text{null}(A) = m_2$ and B_2 is a subset of m_2 rows of B such that $\begin{bmatrix} A & B_2^T \\ B_2 & 0 \end{bmatrix}$ is invertible. Then*

$$\tilde{A}^{-1}(I - B_2^T L^{-1} Z_A^T) = V,$$

where $V = Z_{B_2}(Z_{B_2}^T A Z_{B_2})^{-1} Z_{B_2}^T$.

Proof. Notice that $(I - B_2^T L^{-1} Z_A^T)$ is yet another expression for P_A , the projector onto $\text{range}(A)$ along $\ker(B_2)$. So by Lemma 3.1, we can write

$$(4.1) \quad \tilde{A}^{-1}(I - B_2^T L^{-1} Z_A^T) = \tilde{A}^{-1} A V.$$

Because $A \tilde{A}^{-1} = P_A$ (by Lemma 2.5), $\tilde{A}^{-1} A = (A \tilde{A}^{-1})^T$ is a projector onto $\ker(B_2)$ along $\text{range}(A)$. This means that $\tilde{A}^{-1} A Z_{B_2} = Z_{B_2}$. Therefore,

$$\tilde{A}^{-1} A V = V,$$

which, together with (4.1), completes the proof. \square

For the preconditioner, we set the (2,3)-, (3,2)-, and (3,3)-blocks equal to zero, in common with both inverse formulas. For the (1,3)- and (3,1)-blocks, we use the exact values of the null- A formula. The four blocks in the top left we replace by a block diagonal Schur-complement-like operator, but we use V in place of \tilde{A}^{-1} , yielding the 3×3 block triangular preconditioner:

$$(4.2) \quad \mathcal{P}_{3,T}^{-1} = \begin{bmatrix} V & 0 & Z_A L^{-1} \\ 0 & (B_1 V B_1^T)^{-1} & 0 \\ L^{-1} Z_A^T & 0 & 0 \end{bmatrix}.$$

To show that this matrix is well-defined, we note that $B_1 V B_1^T$ is invertible for any nonsingular \mathcal{K} . Writing

$$B_1 V B_1^T = (B_1 Z_{B_2}) \left(Z_{B_2}^T A Z_{B_2} \right)^{-1} (B_1 Z_{B_2})^T$$

and recalling that $Z_{B_2}^T A Z_{B_2}$ is positive definite, we see that $B_1 V B_1^T$ is positive definite if and only if $B_1 Z_{B_2}$ has full row rank. We prove this by showing that $Z_{B_2}^T B_1^T$ has full column rank. Assume that $Z_{B_2}^T B_1^T x = 0$ for $x \neq 0$. Because B_1^T is full-rank, this must mean that $B_1^T x \in \text{range}(B_2^T)$, which in turn implies that $\text{range}(B_1^T) \cap \text{range}(B_2^T) \neq \emptyset$. This is a contradiction, as it implies that that $\begin{bmatrix} B_1^T & B_2^T \end{bmatrix}$ does not have full column rank, which would mean that \mathcal{K} is singular.

We can now state the eigenvalues of the preconditioned matrix.

THEOREM 4.2. *Suppose that $\text{null}(A) = m_2$ and B_2 is a subset of m_2 rows of B such that $\begin{bmatrix} A & B_2^T \\ B_2 & 0 \end{bmatrix}$ is invertible. Then $\mathcal{P}_{3,T}^{-1} \mathcal{K}$ has eigenvalues $\lambda = 1, \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$, with respective geometric multiplicities $n - m_1 + m_2, m_1$, and m_1 .*

Proof. The preconditioned operator is

$$\mathcal{P}_{3,T}^{-1} \mathcal{K} = \begin{bmatrix} V A + Z_A L^{-1} B_2 & V B_1^T & 0 \\ (B_1 V B_1^T)^{-1} B_1 & 0 & 0 \\ 0 & L^{-1} Z_A^T B_1^T & I \end{bmatrix} = \begin{bmatrix} I & V B_1^T & 0 \\ (B_1 V B_1^T)^{-1} B_1 & 0 & 0 \\ 0 & L^{-1} Z_A^T B_1^T & I \end{bmatrix}.$$

The second equality holds because $VA = (AV)^T = P_A^T$ (Lemma 3.1) and

$$Z_A L^{-1} B_2 = I - (I - B_2^T L^{-1} Z_A^T)^T = I - P_A^T.$$

Also note that we cannot simplify the (3,2)-block of $\mathcal{P}_{3,T}^{-1} \mathcal{K}$ to zero, as we are no longer assuming that B_1 is linearly dependent on A .

We write the eigenvalue equations as

$$(4.3a) \quad x + V B_1^T y_1 = \lambda x;$$

$$(4.3b) \quad (B_1 V B_1^T)^{-1} B_1 x = \lambda y_1;$$

$$(4.3c) \quad L^{-1} Z_A^T B_1^T y_1 + y_2 = \lambda y_2.$$

We immediately find that $\lambda = 1$ is an eigenvalue with multiplicity at least $n - m_1 + m_2$, with eigenvectors of the form

$$\begin{bmatrix} x \\ 0 \\ y_2 \end{bmatrix} \text{ for } x \in \ker(B_1), y_2 \in \mathbb{R}^{m_2}.$$

For $\lambda \neq 1$, we can eliminate y_2 by setting

$$y_2 = \frac{1}{\lambda - 1} L^{-1} Z_A^T B_1^T y_1,$$

by (4.3c). Next, we solve (4.3b) for y_1 and substitute into (4.3a) to give

$$x + \frac{1}{\lambda} V B_1^T (B_1 V B_1^T)^{-1} B_1 x = \lambda x.$$

Note that $V B_1^T (B_1 V B_1^T)^{-1} B_1$ is a rank- m_1 projector onto $\text{range}(V B_1^T)$. Considering x in this range, we obtain

$$x + \frac{1}{\lambda} x = \lambda x,$$

which gives $\lambda = \frac{1 \pm \sqrt{5}}{2}$, each with multiplicity m_1 . Along with the $n - m_1 + m_2$ eigenvalues equal to 1, we have accounted for all eigenvalues. \square

Remark 4.3. There are two ways to compute V . In practice, V needs to be approximated, and depending on the problem in question, one of these formulations may be easier to handle. If we have a way to approximately form and solve the augmented leading block, we could use

$$\tilde{V} \approx (A + B_2^T L^{-1} B_2)^{-1} (I - B_2^T L^{-1} Z_A^T).$$

Alternatively, if a null space of B_2 is available and we can approximately invert $Z_{B_2}^T A Z_{B_2}$, we could instead use

$$\tilde{V} \approx Z_{B_2} (Z_{B_2}^T A Z_{B_2})^{-1} Z_{B_2}^T.$$

Remark 4.4. This preconditioner can also be used if a null space of A is not readily available but a null space of B_2 is. We can use the second strategy of Remark 4.3 to approximate V and replace the (3,1)- and (1,3)-blocks of $\mathcal{P}_{3,T}^{-1}$ by their null- B_2 values, giving the mathematically equivalent but more expensive formulation:

$$\mathcal{P}_{3,T}^{-1} = \begin{bmatrix} V & 0 & (I - VA) B_2^T (B_2 B_2^T)^{-1} \\ 0 & (B_1 V B_1^T)^{-1} & 0 \\ (B_2 B_2^T)^{-1} B_2 (I - AV) & 0 & 0 \end{bmatrix}.$$

Remark 4.5. If $m_1 = 0$ and $m_2 = m$ (i.e., A is maximally rank-deficient), then this preconditioner is actually the exact inverse of \mathcal{K} . See [2] for proof, and for preconditioners derived by approximating the leading block.

5. Discussion and conclusions. We have presented three ideal preconditioners for saddle point systems with a singular leading block. The two block diagonal preconditioners rely on assumptions about the rank structure of the matrix (maximal rank-deficiency or minimal independence), while the block triangular preconditioner does not. While the block diagonal preconditioners may be used when the rank structure assumptions do not hold, we will no longer have a constant number of eigenvalues in the preconditioned operator. We refer to [6] for analysis of $\mathcal{P}_{2,D}$ in the general case. A corresponding investigation for $\mathcal{P}_{3,D}$ is a subject for future work.

In practice, all three preconditioners will be too expensive to apply exactly. All require inverting an augmented leading block and computing a Schur complement or Schur-complement-like expression. Additionally, the 3×3 block preconditioners require identifying a subset of rows of the off-diagonal block that are linearly independent of the leading block, which may be difficult in some cases, and the block triangular preconditioner requires a null space of A or B_2 . Making these preconditioners practical will require effective approximations to the more expensive terms. These approximation strategies will need to be informed by the problem to be solved.

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